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A METHOD OF FINDING ASYMPTOTIC FORMS AT THE COMMON APEX OF ELASTIC WEDGES†

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An efficient method of determining the asymptotic behaviour of stresses and strains in the neighbourhood of the common apex of elastic wedges is proposed. Compared with other known approaches, this method has the merit of using matrices no higher than the second order, regardless of the number of wedges. This is done by exploiting the specific geometry of the problem: the wedges form a chain-type system. Then the use of Mellin transforms and utilization of the geometry reduces the problem to a system of three-point difference equations with matrices no higher than the second order. The determinant of the system is easily computed by the pivotal condensation method. Formulae are given for open and closed systems of elastic wedges, assuming complete bonding and slippage at the contacts, for plane and anti-plane deformations. In the latter case the method includes asymptotic forms for problems described by Laplace's equation.

Further advances in boundary element or finite element methods necessitate taking the corners of interacting wedges into account. Corners are points of intersection of cracks or of boundaries of plane regions with different properties. They may either belong to the outer boundary or be within the body. In the former case (Fig. 1) one has a configuration of "open" type, in the latter (Fig. 2)—one of "closed" type. Sometimes corners are caused by artificial division of the region, in the coupling of boundary and finite elements. In three-dimensional block (granulated) systems there may be such points on the smooth parts of common edges.

A geometric discontinuity, even when generated by a single wedge, requires special treatment if reliable numerical results are desired [1, 2]. Various approaches have been proposed to that end, some of them semi-empirical or "experimental" [2–5]. They are satisfactory for a single wedge (particularly at acute angles). However, the possibility of ensuring correct results in the general case, without resorting to actual asymptotic expansions [6], is highly problematical.

Another, "regular" approach is to use actual asymptotic expansions in the neighbourhood of the corner. If the rigorous theoretical asymptotic forms are known, one can use them to construct special trial functions for the boundary or finite elements contiguous with the corner [7, 8].

This approach comprises two stages. In the first stage one finds rigorous asymptotic relations at the corner. In the second stage these asymptotic relations are substituted into the trial functions for the special, "singular", elements. A simple example of this kind is an end element, which allows for the root asymptotic forms in the neighbourhood of the tip of a crack [9]. It gives a marked increase in the accuracy of numerical results [9, 10].

We shall concentrate on the first stage. Our aim is to give an efficient algorithm for setting up a computer module. The latter should use as input data only the local geometry and properties of the wedges in the neighbourhood of the corner. The output should be the characteristics of the asymptotic forms. These output asymptotic forms, in turn, will serve as input information for modules to generate the trial functions and work with them.

The classical way to obtain asymptotic forms for a wedge of arbitrary angle involves the use of the Mellin transform or, equivalently, some variant of the method of separation of variables [11–31].

Previous publications [11–25] have treated cases of at most three wedges. However, the general case of three or more wedges proves to be more accessible if one has a solution for a single wedge. Indeed, for n wedges, it will suffice to use the Mellin transforms of the solutions for each wedge and couple them together using boundary and contact conditions. This natural approach may be implemented in different ways.

The simplest approach is direct combination of the aforementioned solutions in a system [26, 27]. The characteristic determinant of the system is of order $4n$. It has been written out explicitly for the special case of three wedges ($n = 3$), when it is of order 12 [27].

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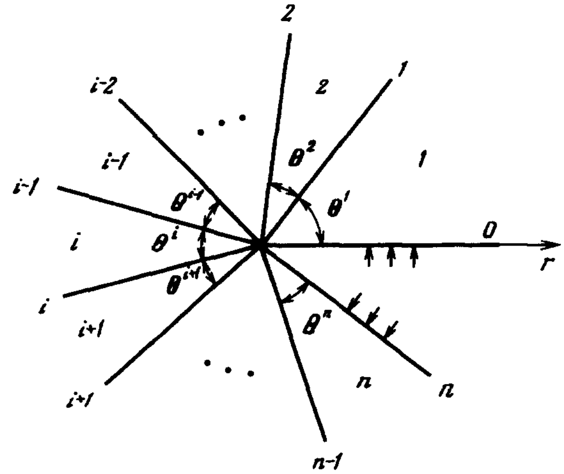

Fig. 1.

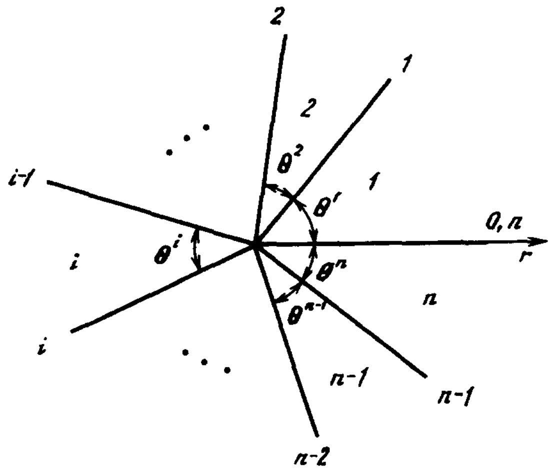

Fig. 2.

In order to avoid matrices and determinants of high orders, one can proceed differently. The idea is to exploit the specific geometry of the problem: these systems of wedges are chain-type systems. This makes it possible to employ three-point difference equations, which can be solved effectively by the pivotal condensation method [32]. The method developed below is based on this approach.

Essentially, unlike previous work, we shall use the special form of the characteristic determinant (of order $4n$). This form takes the special geometry of the problem into consideration. Its treatment by the Gauss elimination method reduces to the pivotal condensation method, and moreover the latter is required for matrices no higher than the second order. Due to the low order of these matrices, as well as the known stability and efficiency of the pivotal condensation method, the method proposed here is superior to previously used approaches. In particular, it is unconditionally more stable than matrix transfer [26] (the reasons for the latter's instability, as well as illustrative examples, may be found, e.g. in [32–34]).

1. INITIAL FORMULAE

We will solve the problem of the asymptotic forms in the neighbourhood of the common corner of a composite wedge in polar coordinates (r, θ) with the origin at the common corner (Figs 1 and 2). To fix our ideas, we will number the wedges in order of increasing angular coordinate θ. (In the case of a closed system the polar axis is placed in an arbitrary wedge.) The angle of the ith wedge will be denoted by Θ^i.

The contacts will also be numbered, assigning the index i to the boundary between the ith and $(i+1)$th wedges. In an open system (Fig. 1) the contact-free (i.e. outer) boundary of the first wedge is assigned the index 0. In a closed system (Fig. 2) the zeroth and nth boundaries coincide.

Quantities referring to the ith wedge will be given the superscript i. The latter's contact with the ith wedge will be indicated by the superscript b, and that with the $(i+1)$th wedge by the subscript t.

The Mellin transform and its inverse are

$$f(s,\theta)=\int_0^\infty f(r,\theta)r^{s-1}dr,\qquad f(r,\theta)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s,\theta)r^{-s}ds \tag{1.1}$$

Throughout, the argument r is that of the function itself, while s labels the Mellin transform. For stresses, as usual [12], we shall use the Mellin transform of the stress multiplied by r^2, but displacements will be multiplied by r, so that, for example

$$\sigma_{\theta\theta}(s,\theta)=\int_0^\infty r^2\sigma_{\theta\theta}(r,\theta)r^{s-1}dr,\qquad u_\theta(s,\theta)=\int_0^\infty ru_\theta(r,\theta)r^{s-1}dr \tag{1.2}$$

This convention will be used wherever we are dealing with a biharmonic problem. (In problems for Laplace's equation, in particular, in the problem of anti-plane deformation, stresses are multiplied by r, but deformations are considered without this factor.)

We shall begin the construction of the system with the treatment of a single wedge. To simplify the notation, we shall not indicate its index, and we shall place the polar axis along the axis of symmetry of the wedge. Denote the angle of the wedge by Θ. Then, using Airy functions and Melon transforms, we obtain

$$\begin{aligned}\sigma_{ij}(s,\theta)&=\sigma_{ij}^s(s,\theta)+\sigma_{ij}^a(s,\theta)\qquad (i,j=\theta,r)\\ u_i(s,\theta)&=u_i^s(s,\theta)+u_i^a(s,\theta)\qquad (i=\theta,r)\end{aligned} \tag{1.3}$$

where the superscript s indicates the symmetric part (relative to the axis of symmetry), and the superscript a the anti-symmetric part; moreover

$$\begin{aligned}\sigma_{\theta\theta}^s(s,\theta)&=A_1^s(s+1)\cos\theta+A_2^s s(s+1)\cos(s+2)\theta\\ \sigma_{r\theta}^s(s,\theta)&=-A_1^s s(s+1)\sin\theta-A_2^s(s+1)(s+2)\sin(s+2)\theta\\ \sigma_{rr}^s(s,\theta)&=-A_1^s s(s+1)\cos\theta-A_2^s(s+1)(s+4)\cos(s+2)\theta\\ u_\theta^s(s,\theta)&=A_1^s\frac{1}{2\mu}s\sin s\theta+A_2^s\left(\frac{1}{2\mu}(s+2)-\frac{4k}{2\mu}\right)\sin(s+2)\theta\\ u_r^s(s,\theta)&=A_1^s\frac{1}{2\mu}s\cos s\theta+A_2^s\left(\frac{1}{2\mu}s+\frac{4k}{2\mu}\right)\cos(s+2)\theta\end{aligned} \tag{1.4}$$

while for the anti-symmetric part we make the following replacements in formulae (1.4): s by a, $\cos(\cdot)$ by $\sin(\cdot)$ and $\sin(\cdot)$ by $-\cos(\cdot)$; $A_1^s, A_2^s, A_1^a, A_2^a$ are coefficients which depend only on s, μ is the shear modulus, $k=1-\nu$ in the plane state of strain and $k=1/(1+\nu)$ in the plane stressed state (k is related to the Muskhelishvili parameter κ [35] by the formula $\nu=4k-1$; for Bogy's parameter m [21] we have $m=4k$).

The asymptotic behaviour of the stresses and displacements at the apex of the wedge is completely defined by the poles of $A_1^s, A_2^s, A_1^a, A_2^a$ as functions of the parameter s. The problem is to determine these poles.

We first write out relations expressing the displacements $\mathbf{u}_t$ and $\mathbf{u}_b$ on the wedge faces in terms of the surface tractions on them. Using (1.3) and (1.4), we obtain a system of equations

$$\begin{aligned}\mathbf{u}_t&=\mathbf{R}_{tt}\mathbf{p}_t+\mathbf{R}_{tb}\mathbf{p}_b\\ \mathbf{u}_b&=\mathbf{R}_{bt}\mathbf{p}_t+\mathbf{R}_{bb}\mathbf{p}_b\end{aligned} \tag{1.5}$$

$$\mathbf{p}_t=\begin{Vmatrix}\sigma_{\theta\theta}(s,\Theta/2)\\ \sigma_{r\theta}(s,\Theta/2)\end{Vmatrix},\qquad \mathbf{p}_b=\begin{Vmatrix}\sigma_{\theta\theta}(s,-\Theta/2)\\ \sigma_{r\theta}(s,-\Theta/2)\end{Vmatrix} \tag{1.6}$$

$$\mathbf{u}_t=\begin{Vmatrix}u_\theta(s,\Theta/2)\\ u_r(s,\Theta/2)\end{Vmatrix},\qquad \mathbf{u}_b=\begin{Vmatrix}u_\theta(s,-\Theta/2)\\ u_r(s,-\Theta/2)\end{Vmatrix} \tag{1.7}$$

$$\begin{aligned}\mathbf{R}_{tt}&=\frac{1}{2}(\mathbf{R}^s+\mathbf{R}^a),\qquad &\mathbf{R}_{tb}&=-\frac{1}{2}(\mathbf{R}^s-\mathbf{R}^a)_1\\ \mathbf{R}_{tb}&=\frac{1}{2}(\mathbf{R}^s-\mathbf{R}^a)',\qquad &\mathbf{R}_{bb}&=-\frac{1}{2}(\mathbf{R}^s+\mathbf{R}^a)_1'\end{aligned} \tag{1.8}$$

$$\mathbf{R}^s = \frac{1}{s+1}\frac{1}{T^s}\frac{1}{2\mu}\begin{Vmatrix} ka_- & -T^s + kb_+ \\ T^s + kb_- & -ka_+ \end{Vmatrix} \tag{1.9}$$

$$\mathbf{R}^a = \frac{1}{s+1}\frac{1}{T^a}\frac{1}{2\mu}\begin{Vmatrix} ka_+ & -T^a + kb_- \\ T^a + kb_+ & -ka_- \end{Vmatrix} \tag{1.10}$$

$$T^s = (s+1)\sin\Theta + \sin(s+1)\Theta, \qquad T^a = (s+1)\sin\Theta - \sin(s+1)\Theta \tag{1.11}$$

$$a_\pm = 2(\cos\Theta \pm \cos(s+1)\Theta), \qquad b_\pm = 2(\sin\Theta \pm \sin(s+1)\Theta) \tag{1.12}$$

The subscript 1 means that the first column in the matrix has been multiplied by –1; a prime means that the first row has been multiplied by –1.

Formulae (1.5) hold for each wedge. Their use for a multi-wedge requires the use of contact conditions. One such condition is the continuity of forces across contacts. At the ith contact we have

$$\mathbf{p}_b^{i+1} = \mathbf{p}_t^i = \mathbf{p}^i \tag{1.13}$$

Another condition may be a given discontinuity of displacements

$$\Delta\mathbf{u}^i = \mathbf{u}_b^{i+1} - \mathbf{u}_t^i = \Delta\mathbf{u}_0^i \tag{1.14}$$

A special case of this condition with $\Delta\mathbf{u}_0^i = \mathbf{0}$ is the full bonding condition. To simplify our argument we shall assume that the functions $\Delta\mathbf{u}_0^i(s)$ have no poles in any finite region of the plane S.

Formulae (1.5), the contact conditions (1.13) and (1.14), and the boundary conditions (for open systems) enable us to construct second-order three-point difference equations for the two-dimensional vectors $\mathbf{p}^i$. The form of the system of equations and the boundary conditions depends on the type of problem. The cases of an open system (Fig. 1) and a closed system (Fig. 2) will therefore be treated separately.

2. OPEN SYSTEM

When there are outer boundaries, we obtain a system of difference equations of the following form [32]

$$\mathbf{A}^i\mathbf{p}^{i-1} - \mathbf{C}^i\mathbf{p}^i + \mathbf{B}^i\mathbf{p}^{i+1} + \mathbf{F}^i = \mathbf{0} \qquad (i = 1, \ldots, n-1) \tag{2.1}$$

where

$$\mathbf{A}^i = -\mathbf{R}_{tb}^i, \quad \mathbf{C}^i = \mathbf{R}_{tt}^i - \mathbf{R}_{bb}^{i+1}, \quad \mathbf{B}^i = \mathbf{R}_{bt}^i, \quad \mathbf{F}^i = -\Delta\mathbf{u}_0^i \tag{2.2}$$

To solve this system, we use the conditions

$$\begin{aligned} -\mathbf{C}^0\mathbf{p}^0 + \mathbf{B}^0\mathbf{p}^1 + \mathbf{F}^0 &= \mathbf{0} \\ \mathbf{A}^n\mathbf{p}^{n-1} - \mathbf{C}^n\mathbf{p}^n + \mathbf{F}^n &= \mathbf{0} \end{aligned} \tag{2.3}$$

These relations exhaust the main types of boundary conditions. Indeed, if the tractions $\mathbf{p}^0$, are prescribed at the zeroth ($i = 0$) boundary, we must write

$$\mathbf{C}^0 = \mathbf{I} = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad \mathbf{B}^0 = \mathbf{0}, \quad \mathbf{F}^0 = \mathbf{p}^0$$

For prescribed displacements $\Delta\mathbf{u}^0 = \mathbf{u}^0$, we have

$$\mathbf{C}^0 = -\mathbf{R}_{bb}^1, \quad \mathbf{B}^0 = \mathbf{R}_{bt}^1, \quad \mathbf{F}^0 = -\mathbf{u}^0$$

Similarly, for the last boundary ($i = n$), with prescribed tractions $\mathbf{p}^n$

$$\mathbf{A}^n = \mathbf{0}, \quad \mathbf{C}^n = \mathbf{I}, \quad \mathbf{F}^n = \mathbf{p}^n$$

and for prescribed displacements $\Delta\mathbf{u}^n = -\mathbf{u}^n$

$$\mathbf{A}^n = -\mathbf{R}^n_{tb}, \quad \mathbf{C}^n = \mathbf{R}^n_{tt}, \quad \mathbf{F}^n = \mathbf{u}^n$$

Problem (2.1), (2.3) is solved by matrix pivotal condensation [36]. In the direct pivotal condensation one successively finds the matrices $\boldsymbol{\alpha}^i$ and the vectors $\boldsymbol{\beta}^i$

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = (\mathbf{C}^0)^{-1}\mathbf{B}^0, \quad \boldsymbol{\beta}^1 = (\mathbf{C}^0)^{-1}\mathbf{F}^0 \\
&\boldsymbol{\alpha}^{i+1} = (\mathbf{C}^i - \mathbf{A}^i\boldsymbol{\alpha}^i)^{-1}\mathbf{B}^i \qquad (i = 1,\ldots,n-1) \\
&\boldsymbol{\beta}^{i+1} = (\mathbf{C}^i - \mathbf{A}^i\boldsymbol{\alpha}^i)^{-1}(\mathbf{F}^i + \mathbf{A}^i\boldsymbol{\beta}^i) \qquad (i = 1,\ldots,n)
\end{aligned} \tag{2.4}$$

Then, in the retrograde pivotal condensation, one determines the transforms of the tractions at the contacts

$$\mathbf{p}^n = \boldsymbol{\beta}^{n+1}, \quad \mathbf{p}^i = \boldsymbol{\alpha}^{i+1}\mathbf{p}^{i+1} + \boldsymbol{\beta}^{i+1} \qquad (i = n-1,\ldots,1) \tag{2.5}$$

Taking the inverse Mellin transforms of the resulting expressions $\mathbf{p}^i$, one obtains a solution to the initial problem. By the residue theorem, the principal and following terms of its asymptotic expansion are determined by the poles of the integrands. These poles are uniquely defined by the determinant D of system (2.1), (2.3). In view of its specific structure, D may be expressed in terms of the second-order sweeping matrices $\boldsymbol{\alpha}^i$ ($i = 1, \ldots, n$). The formula is

$$D(s) = \det \prod_{i=0}^{n} (\mathbf{A}^i\boldsymbol{\alpha}^i - \mathbf{C}^i) = \prod_{i=0}^{n} \det(\mathbf{A}^i\boldsymbol{\alpha}^i - \mathbf{C}^i) \tag{2.6}$$

where $\boldsymbol{\alpha}^0 = \mathbf{0}$.

It can be shown (we shall not go into details) that the roots of the equation

$$D(s) = 0 \tag{2.7}$$

determine the asymptotic behaviour of the stresses (and displacements) not only at the contacts but also inside the wedges. The asymptotic relationships are the same in all wedges.

The roots of Eq. (2.7) are symmetric about the real axis and the straight line $\operatorname{Re} s = -1$. Therefore, if $s = a + ib$ ($a > -1$, $b > 0$) is a root of the equation, so are the numbers $a - ib$, $-a + ib - 2$, $-a - ib - 2$. They determine the asymptotic forms as $r \to 0$ and $r \to \infty$.

The asymptotic behaviour of the stresses as $r \to 0$ is determined by the roots of (2.7) that lie to the left of the line $\operatorname{Re} s = -1$. For $r \to \infty$, the significant roots are those to the right of the line. (We have taken into account here that the transforms of the stresses and the forces are determined, according to (1.2), with the stresses multiplied by r^2.)

Let us consider the asymptotic forms as $r \to 0$. This means considering the roots of (2.7) with $\operatorname{Re} s < -1$. For complex roots, we need consider only those for which $\operatorname{Im} s > 0$. Number all such roots in order of non-increasing real part. Conjugate roots, for which $\operatorname{Im} s_i < 0$, guarantee that the inverse transform is real for the tractions and all the stress components. As a result, if $r < 1$ the components of the physical stresses (i.e. not multiplied by r^2) are

$$\sigma = \operatorname{Re} \sum_{k=1}^{\infty} C_k r^{-s_k - 2} \qquad (r < 1) \tag{2.8}$$

where $\operatorname{Re} s_k < -1$; $\operatorname{Im} s_k > 0$, $k = 1, 2, \ldots$. The representation (2.8) takes it for granted that all the roots are simple; multiple roots lead to terms with a factor $\ln r$.

The displacements admit of similar asymptotic expansions

$$u = \operatorname{Re} \sum_{k=1}^{\infty} B_k r^{-s_k - 1} \qquad (r < 1) \tag{2.9}$$

where the roots s_k are the same as in (2.8).

In practical work one uses the first or the first few terms of (2.8) and (2.9). The only roots leading to singular stresses are those in the strip $-2 < \operatorname{Re} s < -1$. We know [14, 20, 28–30] that such roots do not always exist. But in the case when $s_1 < -2$ the use of the first terms of the asymptotic representations (2.8) and (2.9) is desirable to ensure accurate and stable numerical results.

3. CLOSED SYSTEMS

Systems of closed type (Fig. 2) are quite common for block or granulated structures, and also for welded joints. Important numerical results for two and three rigidly coupled wedges may be found in [31]. General systems of this type may be considered, subject to only slight modifications in the reasoning of Section 2. The use of (1.5), (1.13) and (1.14) produces the following system of difference equations

$$\mathbf{A}^0\mathbf{p}^{n-1} - \mathbf{C}^0\mathbf{p}^0 + \mathbf{B}^0\mathbf{p}^1 + \mathbf{F}^0 = \mathbf{0} \qquad (i=0) \tag{3.1}$$

$$\mathbf{A}^i\mathbf{p}^{i-1} - \mathbf{C}^i\mathbf{p}^i + \mathbf{B}^i\mathbf{p}^{i+1} + \mathbf{F}^i = \mathbf{0} \qquad (i=1,\ldots,n-1) \tag{3.2}$$

$$\mathbf{p}^n = \mathbf{p}^0 \tag{3.3}$$

where

$$\mathbf{A}^0 = \mathbf{A}^n = -\mathbf{R}^n_{tb}, \quad \mathbf{C}^0 = \mathbf{C}^n = \mathbf{R}^n_{tt} - \mathbf{R}^1_{bb}, \quad \mathbf{B}^0 = \mathbf{B}^n = \mathbf{R}^1_{bt}, \quad \mathbf{F}^0 = \mathbf{F}^n = -\Delta\mathbf{u}^n_0$$

For $i = 1, \ldots, n-1$ the coefficients $\mathbf{A}^i$, $\mathbf{C}^i$, $\mathbf{B}^i$ and $\mathbf{F}^i$ are defined by formulae (2.2). Condition (3.3) is included so as not to exclude $\mathbf{p}^n$ from (3.2) at $i = n-1$; this enables us to maintain the form of (3.2) when $i = 1, n-1$.

To solve Eqs (3.1)–(3.3) we use a version of the matrix pivotal condensation method—cyclic matrix pivotal condensation. This method is based on an analogous method for scalar systems [36]. In the direct pivotal condensation we have

$$\begin{aligned}
&\boldsymbol{\alpha}^1 = \mathbf{0}, \quad \boldsymbol{\beta}^1 = \mathbf{0}, \quad \boldsymbol{\gamma}^1 = \mathbf{V}^0 = \mathbf{I} \\
&\boldsymbol{\alpha}^{i+1} = (\mathbf{C}^i - \mathbf{A}^i\boldsymbol{\alpha}^i)^{-1}\mathbf{B}^i, \qquad \boldsymbol{\beta}^{i+1} = (\mathbf{C}^i - \mathbf{A}^i\boldsymbol{\alpha}^i)^{-1}(\mathbf{F}^i + \mathbf{A}^i\boldsymbol{\beta}^i) \\
&\boldsymbol{\gamma}^{i+1} = (\mathbf{C}^i - \mathbf{A}^i\boldsymbol{\alpha}^i)^{-1}\mathbf{A}^i\boldsymbol{\gamma}^i \qquad (i=1,\ldots,n-1)
\end{aligned} \tag{3.4}$$

The retrograde pivotal condensation generates the auxiliary quantities

$$\begin{aligned}
&\mathbf{q}^n = \mathbf{0}, \qquad \mathbf{V}^n = \mathbf{I} \\
&\mathbf{q}^i = \boldsymbol{\alpha}^{i+1}\mathbf{q}^{i+1} + \boldsymbol{\beta}^{i+1}, \qquad \mathbf{V}^i = \boldsymbol{\alpha}^{i+1}\mathbf{V}^{i+1} + \boldsymbol{\gamma}^{i+1} \qquad (i=n-1,\ldots,1)
\end{aligned} \tag{3.5}$$

where the vectors are a solution of the inhomogeneous equations with homogeneous boundary conditions

$$\begin{aligned}
&\mathbf{A}^i\mathbf{q}^{i-1} - \mathbf{C}^i\mathbf{q}^i + \mathbf{B}^i\mathbf{q}^{i+1} + \mathbf{F}^i = \mathbf{0} \qquad (i=1,\ldots,n-1) \\
&\mathbf{q}^0 = \mathbf{0}, \qquad \mathbf{q}^{n} = \mathbf{0}
\end{aligned}$$

and the matrices $\mathbf{V}^i$ solve the homogeneous problem with inhomogeneous boundary conditions

$$\begin{aligned}
&\mathbf{A}^i\mathbf{V}^{i-1} - \mathbf{C}^i\mathbf{V}^i + \mathbf{B}^i\mathbf{V}^{i+1} = \mathbf{0} \qquad (i=1,\ldots,n-1) \\
&\mathbf{V}^0 = \mathbf{I}, \qquad \mathbf{V}^n = \mathbf{I}
\end{aligned}$$

($\mathbf{I}$ is the 2×2 identify matrix).

Then the solution of Eqs (3.1)–(3.3) is given by the formulae

$$\begin{aligned}
&\mathbf{p}^0 = \mathbf{p}^n = -(\mathbf{A}^0\mathbf{V}^{n-1} - \mathbf{C}^0 + \mathbf{B}^0\mathbf{V}^1)^{-1}(\mathbf{F}^0 + \mathbf{A}^0\mathbf{q}^{n-1} + \mathbf{B}^0\mathbf{q}^1) \\
&\mathbf{p}^i = \mathbf{q}^i + \mathbf{V}^i\mathbf{p}^0 \qquad (i=1,\ldots,n-1)
\end{aligned} \tag{3.6}$$

The determinant of system (3.1)–(3.3) uniquely defines the asymptotic behaviour of the stresses and displacements as $r \to 0$ and $r \to \infty$. In the case being treated, as follows from (3.4)–(3.6), the determinant has the form

$$D = \det(\mathbf{A}^0\mathbf{V}^{n-1} - \mathbf{C}^0 + \mathbf{B}^0\mathbf{V}^1) \prod_{i=1}^{n-1} \det(\mathbf{A}^i\boldsymbol{\alpha}^i - \mathbf{C}^i) \tag{3.7}$$

As in an open system, the determinant is expressed in terms of 2×2 determinants. The roots of the equation $D = 0$ determine the exponents s_k in the representations (2.8) and (2.9) for the stresses and displacements.

4. THE CASE OF SMOOTH CONTACTS

When there is no friction at the contacts, one has zero shear stresses and prescribed discontinuities of the normal displacements at each contact. One must then retain in (1.5)–(1.7) only the normal components of the vectors $\mathbf{p}_t$, $\mathbf{p}_b$, $\mathbf{u}_t$, $\mathbf{u}_b$. We therefore put

$$p_t^i = \sigma_{\theta\theta}^i(s, \Theta/2), \qquad p_b^i = \sigma_{\theta\theta}^i(s, -\Theta/2)$$
$$u_t^i = u_\theta^i(s, \Theta/2), \qquad u_b^i = u_\theta^i(s, -\Theta/2)$$
$$\Delta u^i = u_b^{i+1} - u_t^i, \qquad \Delta u_0^i = \Delta u_{\theta 0}^i$$

All these quantities are scalars.

Applying (1.5), the contact conditions (1.13) and (1.14) and definitions (1.8)–(1.10) for normal components only, we arrive at formulae of the same form as (2.1) with

$$\begin{aligned} R_{tt} &= -R_{bb} = \frac{1}{2}\frac{k}{2\mu}\frac{1}{s+1}\left(\frac{a_-}{T^s} + \frac{a_+}{T^a}\right) \\ R_{tb} &= -R_{bt} = \frac{1}{2}\frac{k}{2\mu}\frac{1}{s+1}\left(\frac{a_-}{T^s} - \frac{a_+}{T^a}\right) \end{aligned} \tag{4.1}$$

and T^s, T^a, a_+, a_- as defined by (1.11), (1.12).

The subsequent reasoning is the same as in Sections 2 and 3. We arrive at a system of (scalar) equations of the form (2.1), (2.3). The coefficients are defined by formulae (2.2), but now with R_{tt}, R_{tb}, R_{bt}, R_{bb} given by the new formulae (4.1). System (2.1), (2.2) is solved by pivotal condensations using formulae (2.4) and (2.5). The roots of its determinant uniquely define the asymptotic relations. The only difference is that now all the coefficients A^i, C^i, B^i, α^i are scalars. The $A^i\alpha^i - C^i$ $(i = 0, \ldots, n)$ in (2.6) is a scalar, and the characteristic determinant (2.6) may be written

$$D = \prod_{i=0}^{n} (A^i\alpha^i - C^i) \tag{4.2}$$

where $\alpha^0 = 0$.

In the case of a closed system we obtain from (3.7), in exactly the same way

$$D = (A^0V^{n-1} - C^0 + B^0V^1) \prod_{i=1}^{n-1} (A^i\alpha^i - C^i) \tag{4.3}$$

where now V^i $(i = 0, \ldots, n)$ are also scalars; when computing them one assumes that $V^0 = V^n = 1$.

5. ANTI-PLANE DEFORMATION

The case of anti-plane deformations is not only interesting in itself (as representing a special kind of deformation). Its asymptotic forms remain valid in a more general situation: they are partial asymptotic forms for three-dimensional blocks around common points on smooth parts of their interacting edges. In addition, the consideration of anti-plane deformations provides a sample treatment of a typical harmonic problem.

In anti-plane deformation the displacement u_z is a harmonic function. Taking the Mellin transform (1.1) of the displacement of Laplace's equation, we obtain the general solution of the transformed equation

$$u_z(s,\theta) = A_1 \cos s\theta + A_2 \sin s\theta$$

Then the transforms of the stresses, multiplied by r (and not by r^2 as previously), are

$$\sigma_{\theta z}(s,\theta) = \int_0^\infty r\sigma_{\theta z}(r,\theta) r^{s-1} dr = -\mu s(A_1 \sin s\theta - A_2 \cos s\theta)$$

$$\sigma_{rz}(s,\theta) = \int_0^\infty r\sigma_{rz}(r,\theta) r^{s-1} dr = -\mu s(A_1 \cos s\theta + A_2 \sin s\theta)$$

In the case at hand, formulae (1.5) are scalar, with

$$p_t = \sigma_{\theta z}(s, \Theta/2), \qquad p_b = \sigma_{\theta z}(s, -\Theta/2)$$
$$u_t = u_z(s, \Theta/2), \qquad u_b = u_z(s, -\Theta/2)$$
$$R_{tt} = -R_{bb} = -\frac{1}{\mu s}\operatorname{ctg} s\Theta, \quad R_{tb} = -R_{bt} = \frac{1}{\mu s}\frac{1}{\sin s\Theta} \tag{5.1}$$

The rest of the argument remains unchanged. For an open system, applying (1.5), (1.13) and (1.14), we obtain difference equations (2.1) and (2.3), which are now scalar equations. Their coefficients are given by formulae (2.2), and R_{tt}, R_{tb}, R_{bt}, R_{bb} by the new formulae (2.2).

All the subsequent reasoning is the same as in Sections 2 and 3. The only difference is that the formulae for the pivotal condensation coefficients and the determinant D contain only scalar quantities. For D we obtain a formula of type (4.2) for open systems and (4.3) for closed systems. Once again, the equations $D = 0$ uniquely defines the asymptotic forms. Now, however, the roots are symmetrical about the $\operatorname{Re} s = 0$ axis. Then, if the roots are simple, we replace (2.8) and (2.9) by

$$\sigma = \operatorname{Re}\sum_{k=1}^{\infty} C_k r^{-s_k-1}, \qquad u_z = \operatorname{Re}\sum_{k=1}^{\infty} B_k r^{-s_k} \quad (r < 1)$$

where $\operatorname{Re} s_k < 0$, $\operatorname{Im} s_k \geqslant 0$. This slight difference is due to the fact that in this case the stresses are multiplied by r (instead of r^2 in (1.2)) and the displacements u_z are not multiplied by r (unlike (1.2)).

The expressions for the characteristic determinants are readily obtained in an alternative form. For this purpose one can use, instead of formulae (1.5), which are solved for the displacements, formulae that are solved for the tractions. The final formulae are equivalent to those presented above, but more complicated.

The main application of this method may be to develop a universal program module to construct trial functions for special "singular" elements in boundary element and finite element methods. Only with the help of such elements can one substantially improve the accuracy and reliability of computations for block (granulated) media.

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